Width Parameters on Even-Hole-Free Graphs

Ni Luh Dewi Sintiari

LIP, ENS de Lyon

Supervised by Nicolas Trotignon

June 29, 2021

Outlines

Title: *Width Parameters on Even-Hole-Free Graphs*

Chapter 1: Introduction

Graphs

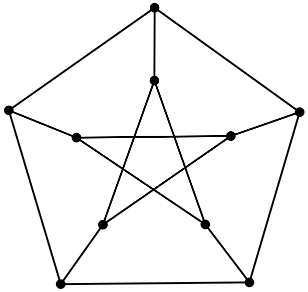


Figure: A graph G

- Vertices or nodes (denoted by $V(G)$)
- Edges (denoted by $E(G)$)

A motivating example

- Graphs are used to model pairwise relations between objects.

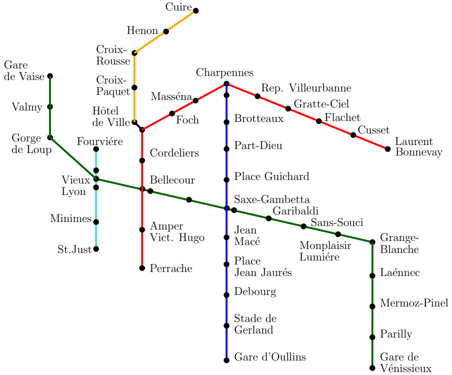
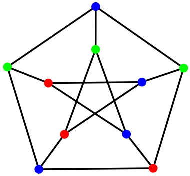


Figure: Graph representation of Lyon subway network

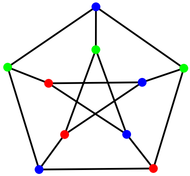
Some well-known graph optimization problems (*NP-Complete*)



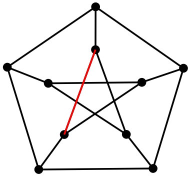
coloring
chromatic number : χ

- **Coloring:** Assignment of colors to the vertices, no adjacent vertices receive the same color → minimize the number of colors

Some well-known graph optimization problems (*NP-Complete*)



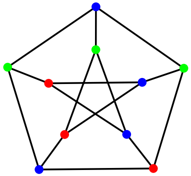
coloring
chromatic number : χ



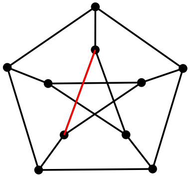
maximum clique
clique number : ω

- **Coloring:** Assignment of colors to the vertices, no adjacent vertices receive the same color $\rightarrow$ minimize the number of colors
- **Maximum clique:** Finding set of pairwise adjacent vertices with maximum cardinality

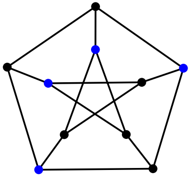
Some well-known graph optimization problems (*NP-Complete*)



coloring
chromatic number : χ



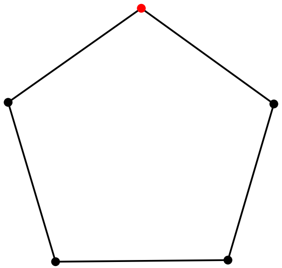
maximum clique
clique number : ω



max independent set
independent set number: α

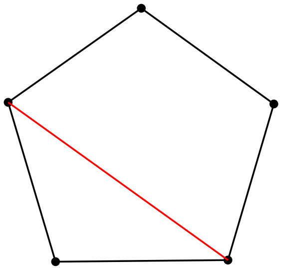
- **Coloring:** Assignment of colors to the vertices, no adjacent vertices receive the same color $\rightarrow$ minimize the number of colors
- **Maximum clique:** Finding set of pairwise adjacent vertices with maximum cardinality
- **Maximum independent set:** Finding set of pairwise non-adjacent vertices with maximum cardinality

Terminology: even hole



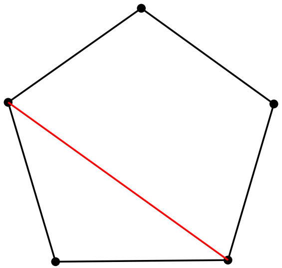
cycle

Terminology: even hole

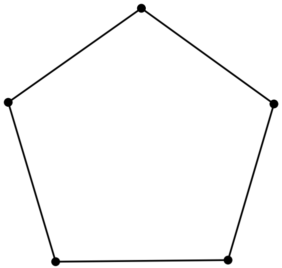


cycle

Terminology: even hole

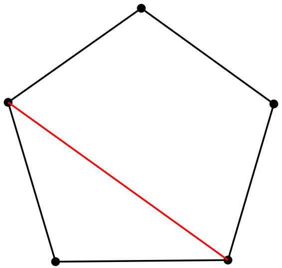


cycle

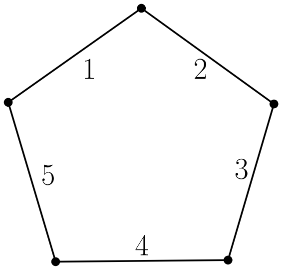


chordless cycle
(*hole* if it has length ≥ 4)

Terminology: even hole

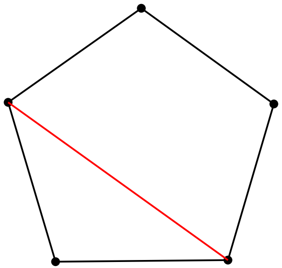


cycle

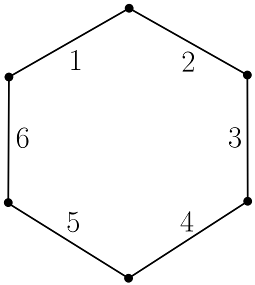


odd hole

Terminology: even hole



cycle



even hole

Terminology: induced subgraph, even-hole-free graphs

- H is an induced subgraph of G if H can be obtained from G by *deleting vertices* (denoted by $H \subseteq_{\text{ind}} G$)

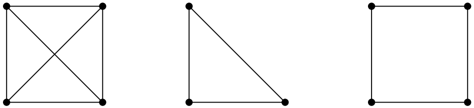


Figure: A graph, an induced subgraph, and a non-induced subgraph

Terminology: induced subgraph, even-hole-free graphs

- H is an induced subgraph of G if H can be obtained from G by *deleting vertices* (denoted by $H \subseteq_{\text{ind}} G$)

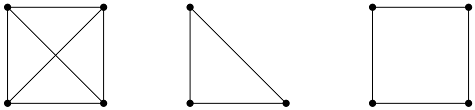


Figure: A graph, an induced subgraph, and a non-induced subgraph

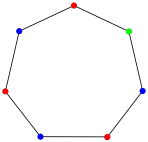
- G is H-free if no induced subgraph of G is isomorphic to H
- When $\mathcal{F}$ is a family of graphs, $\mathcal{F}$-free means H-free, $\forall H \in \mathcal{F}$

Even-hole-free (EHF): the graph does not contain even hole as an induced subgraph

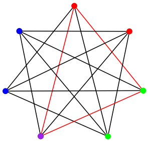
Motivation: relation to perfect graphs

Perfect graphs:

- Every graph G satisfies $\chi(G) \geq \omega(G)$
- G is perfect if $\forall\ H \subseteq_{\text{ind}} G$, $\chi(H) = \omega(H)$



$C_{2k+1}, k = 3,\ \chi(G) = 3,\ \omega(G) = 2$



$\overline{C_{2k+1}},\ \chi(G) = k + 1,\ \omega(G) = k$

an antihole $\overline{C_k}$: *the "complement" of a hole C_k*

- Strong Perfect Graph Conjecture (Berge, 1961):

  G is perfect iff G is (odd hole, odd antihole)-free

  (proved by Chudnovsky, Robertson, Seymour, Thomas (2002))

Motivation: perfect graphs versus even-hole-free graphs

Dichotomy of the two classes:

- EHF graphs are (even hole, even antihole length ≥ 6)-free

Comparison of the decomposition theorems

Decomposition theorem: If G belongs to $\mathcal{C}$ then G is either "basic" or G has some particular cutset.

	EHF graphs *	Perfect graphs†
Basic graphs	cliques, holes, long pyramids, nontrivial basic	bipartite, $\overline{\text{bipartite}}$, L(bipartite), $\overline{\text{L(bipartite)}}$, doubled graphs
Cutsets	2-join, star cutset	2-join, $\overline{\text{2-join}}$ homogeneous pair, balanced skew partition

*Ref: Conforti, Cornuéjols, Kapoor, Vušković (2002)
†Ref: Chudnovsky, Robertson, Seymour, Thomas (2002)

Motivation: perfect graphs versus even-hole-free graphs

	EHF graphs	Perfect graphs
Structure	"simpler"	more complex
Maximum clique	easy	easy
Coloring	?	easy
Maximum independent set	?	easy

"Easy" means the complexity is polynomial

- Goal of study: to have better understanding of the structure of even-hole-free graphs

Chapter 2: Survey on Tree-width

Tree-width and chordalization

Tree-width: *a parameter measuring how far a graph from being a tree*

$$\mathrm{tw}(G) = \min_{H \text{ chordalization of } G} \{\omega(H) - 1\}$$

- Chordal graphs are graphs possessing no hole
- A chordalization of G is a graph H obtained by adding edges to G, such that H is chordal

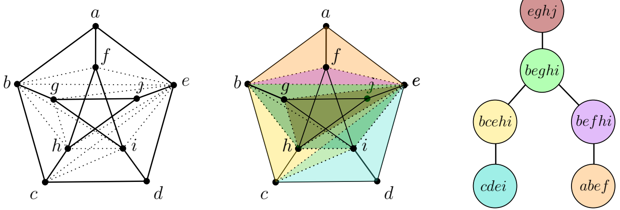


Figure: A chordalization of a graph and its tree-like structure

The use of tree-width

Theorem (Courcelle (1990))

Every graph property definable in the monadic second-order logic of graphs can be decided in linear time on graphs of bounded tree-width.

Some graph problems expressible in MSO:

- coloring, maximum independent set, maximum clique

The use of tree-width

Theorem (Courcelle (1990))

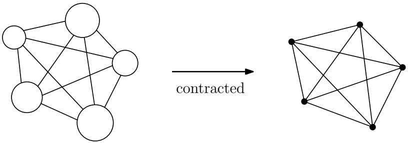
Every graph property definable in the monadic second-order logic of graphs can be decided in linear time on graphs of bounded tree-width.

Some graph problems expressible in MSO:

- coloring, maximum independent set, maximum clique

Sufficient conditions for graphs with high tree-width:

- large ω
- big clique minor



Relation between width parameters

Lemma (Corneil, Rotics (2005) and Oum, Seymour (2006))

For every graph G, the followings hold:

- $\text{rw}(G) \leq \text{cw}(G) \leq 2^{\text{rw}(G)+1}$;
- $\text{cw}(G) \leq 3 \cdot 2^{\text{tw}(G)-1}$;
- $\text{tw}(G) \leq \text{pw}(G)$.

Notation: rw: *rank-width*, cw: *clique-width*, tw: *tree-width*, pw: *path-width*

Survey: *bounded* tree-width EHF graphs

Remark: *in general, the tree-width of even-hole-free graphs is unbounded*

- *Planar* EHF → $tw \leq 49$ [Silva, da Silva, Sales (2010)]
- K_3*-free* EHF → $tw \leq 5$ [Cameron, da Silva, Huang, Vušković (2018)]
- *Pan-free* EHF → $tw \leq 1.5\omega(G) - 1$ [Cameron, Chaplick, Hoàng (2015)]
- *Cap-free* EHF → $tw \leq 6\omega(G) - 1$ [Cameron, da Silva, Huang, Vušković (2018)]

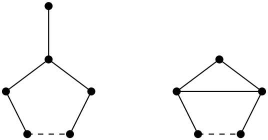
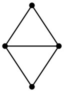


Figure: Pan and cap

Survey: *unbounded* tree-width EHF graphs

Diamond-free EHF → unbounded *rank-width* (*stronger*)
[Adler, Le, Müller, Radovanović, Trotignon, Vušković (2017)]



diamond

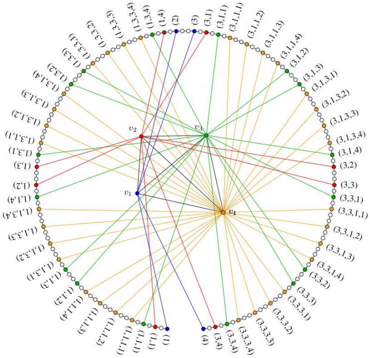
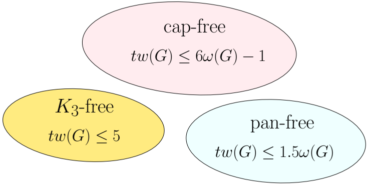


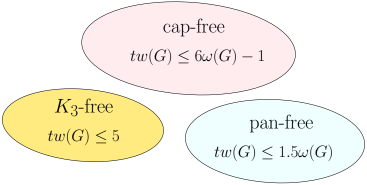
Figure: A diamond-free EHF graph without clique cutset with large rank-width

Problem statement



- All examples of EHF graphs with unbounded width contain large cliques

Problem statement

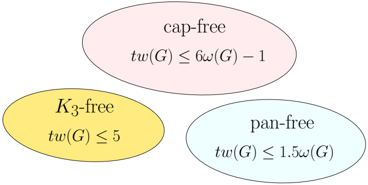


- All examples of EHF graphs with unbounded width contain large cliques

Problem (Cameron, Chaplick, Hoàng (2018))

Let G be even-hole-free, is $tw(G) \leq f(\omega(G))$?

Problem statement



- All examples of EHF graphs with unbounded width contain large cliques

Problem (Cameron, Chaplick, Hoàng (2018))

Let G be even-hole-free, is $tw(G) \leq f(\omega(G))$?

- No, we prove that there are EHF graphs with small ω, but high tree-width
- We study even-hole-free graphs with no K_4

Chapter 3: Layered Wheel

Truemper configurations

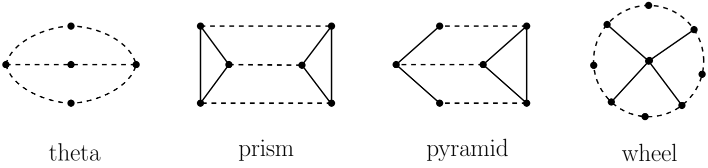


Figure: Truemper configurations; dashed lines represent paths of length at least 1

Truemper configurations

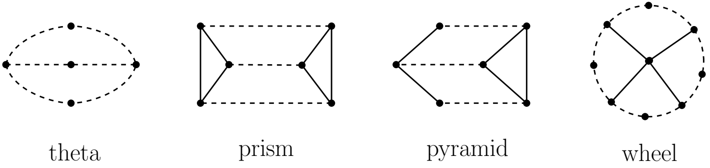
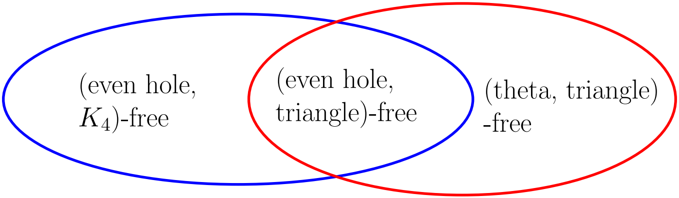


Figure: Truemper configurations; dashed lines represent paths of length at least 1

- They appear in the decomposition theorems of graphs in the classes

	EHF graphs	Perfect graphs
Theta	×	✓
Prism	×	✓
Pyramid	✓	×
Wheel	(no even wheel)	(no wheel of some kind)

Theta-free graphs (TTF) & even-hole-free graphs (EHF)



	(Even hole, K_4)-free graphs	(Theta, triangle)-free graphs
Theta	×	×
Prism	×	×
Pyramid	×	✓
Wheel	✓	✓

Our main results

We prove that the following classes have *unbounded* tree-width

- (Theta, triangle)-free graphs
- (Even hole, K_4)-free graphs

Layered wheels: family of graphs in the classes with high tree-width

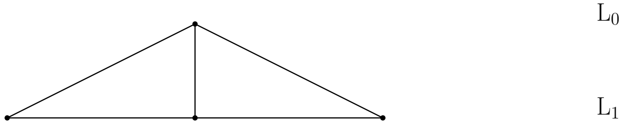
Notation $G_{\ell,k}$

- $\ell \geq 1$ *is the number of layers*
- $k \geq 4$ *is the length of the shortest hole*

Construction: *(theta, triangle)-free* layered wheel

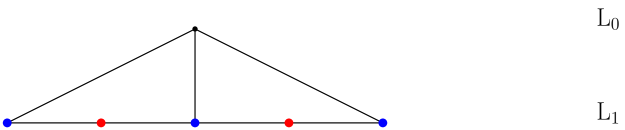
TTF layered wheel $G(\ell, k)$, with $\ell = 2$ and $k = 4$

Construction: *(theta, triangle)-free* layered wheel



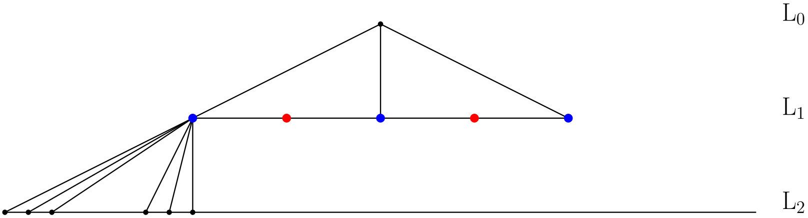
TTF layered wheel $G(\ell, k)$, with $\ell = 2$ and $k = 4$

Construction: *(theta, triangle)-free* layered wheel



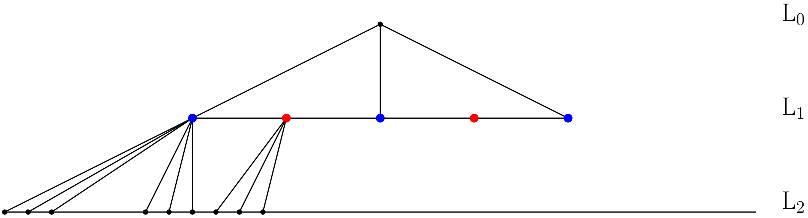
TTF layered wheel $G(\ell, k)$, with $\ell = 2$ and $k = 4$

Construction: *(theta, triangle)-free* layered wheel



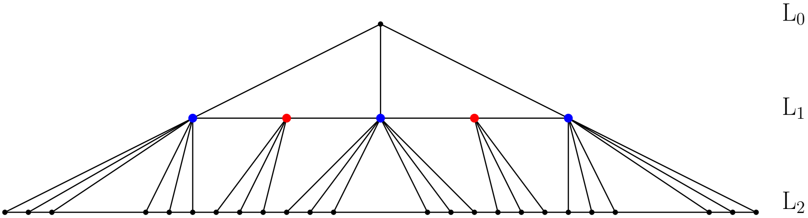
TTF layered wheel $G(\ell, k)$, with $\ell = 2$ and $k = 4$

Construction: *(theta, triangle)-free* layered wheel



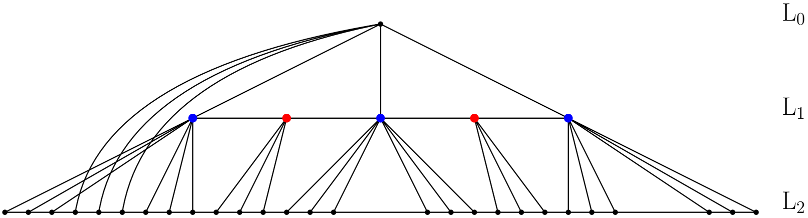
TTF layered wheel $G(\ell, k)$, with $\ell = 2$ and $k = 4$

Construction: *(theta, triangle)-free* layered wheel



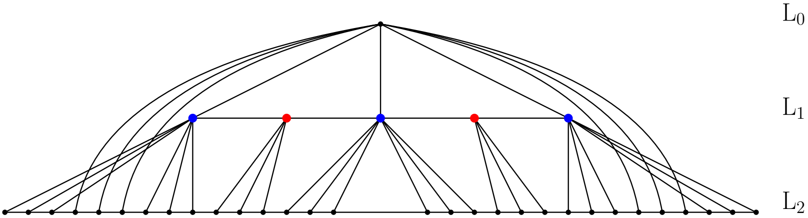
TTF layered wheel $G(\ell, k)$, with $\ell = 2$ and $k = 4$

Construction: *(theta, triangle)-free* layered wheel



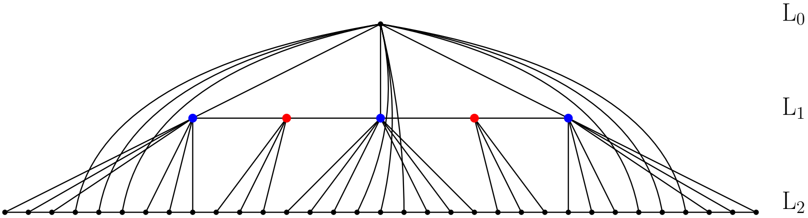
TTF layered wheel $G(\ell, k)$, with $\ell = 2$ and $k = 4$

Construction: *(theta, triangle)-free* layered wheel



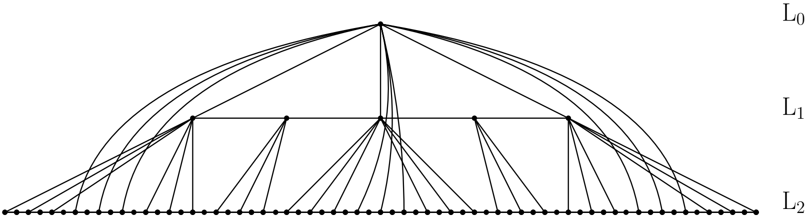
TTF layered wheel $G(\ell, k)$, with $\ell = 2$ and $k = 4$

Construction: *(theta, triangle)-free* layered wheel



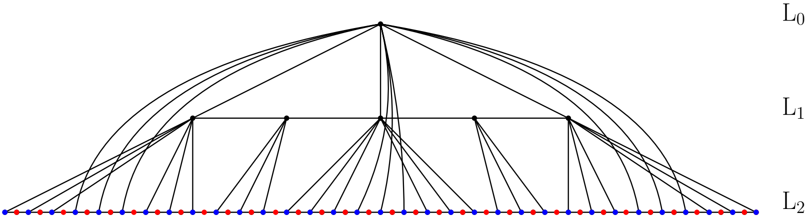
TTF layered wheel $G(\ell, k)$, with $\ell = 2$ and $k = 4$

Construction: *(theta, triangle)-free* layered wheel



TTF layered wheel $G(\ell, k)$, with $\ell = 2$ and $k = 4$

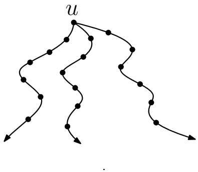
Construction: *(theta, triangle)-free* layered wheel

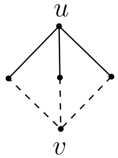


TTF layered wheel $G(\ell, k)$, with $\ell = 2$ and $k = 4$

Sketch of proof

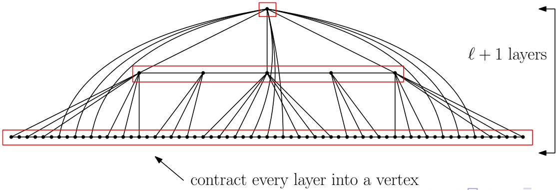
- $G_{\ell,k}$ is full of subdivision of claws, but it is theta-free





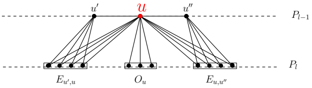
theta

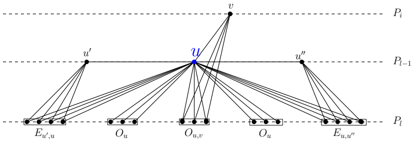
- $tw(G_{\ell,k}) \geq \ell$, because $G_{\ell,k}$ contains big clique minor

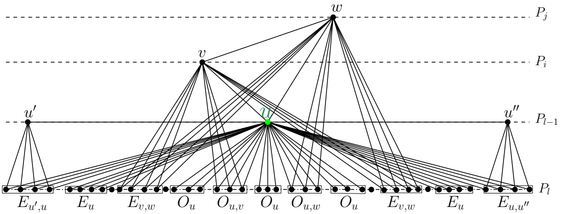


Construction: *(even hole, K_4)-free* layered wheel

- The first two layers are similar to TTF-layered-wheel







Logarithmic bound on the tree-width of layered wheels

The tree-width of layered wheel is still bounded

Theorem (Sintiari, Trotignon (2019))

$tw(G_{\ell,k}) = O(\log(|V(G_{\ell,k})|))$

Key of proof:

1. To reach $tw(G_{\ell,k}) \geq \ell$, it must be $|V(G_{\ell,k})| \geq 3^{\ell}$ vertices.
2. Upper bound: $tw(G_{\ell,k}) \leq 2\ell$.

Summary of results

Theorem (Sintiari, Trotignon (2019))

$\forall \ell \geq 1, k \geq 4$ integers, $\exists$ a graph $G_{\ell,k}$ s.t.

- *$G_{\ell,k}$ is theta-free;*
- *every hole in $G_{\ell,k}$ has length $\geq k$;*
- *$\ell \leq tw(G_{\ell,k}) \leq c \cdot \log(|V(G_{\ell,k})|)$, for some constant c.*

Summary of results

Theorem (Sintiari, Trotignon (2019))

$\forall \ell \geq 1, k \geq 4$ integers, $\exists$ a graph $G_{\ell,k}$ s.t.

- *$G_{\ell,k}$ is theta-free;*
- *every hole in $G_{\ell,k}$ has length $\geq k$;*
- *$\ell \leq tw(G_{\ell,k}) \leq c \cdot \log(|V(G_{\ell,k})|)$, for some constant c.*

Theorem (Sintiari, Trotignon (2019))

$\forall \ell \geq 1, k \geq 4$ integers, $\exists$ a graph $G_{\ell,k}$ s.t.

- *$G_{\ell,k}$ is (even hole, K_4, pyramid)-free;*
- *every hole in $G_{\ell,k}$ has length $\geq k$;*
- *$\ell \leq tw(G_{\ell,k}) \leq c \cdot \log(|V(G_{\ell,k})|)$, for some constant c.*

Chapter 4: Bounds on Tree-width

Motivation: the logarithmic conjecture

Conjecture (*Logarithmic tree-width*; Sintiari, Trotignon (2019))

$\exists c$ constant s.t. $\forall$ (even hole, K_4)-free graph G,
$tw(G) \leq c \log |V(G)|$.

If the conjecture is true, then many graph optimization problems are polynomial-time solvable.

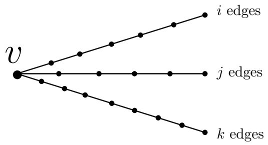
Theorem (Bodlaender (1988))

$\forall G$, given a tree decomposition of width w, the Weighted Maximum Independent Set can be solved in time $\mathcal{O}(2^w \cdot n)$.

Excluding $S_{i,j,k}$ & implication on tree-width

For all fixed non-negative integers i, j, k, t, the following classes have bounded tree-width:

- (theta, triangle, $S_{i,j,k}$)-free graphs
- (even hole, pyramid, K_t, $S_{i,j,k}$)-free graphs



$S_{i,j,k}$

Why excluding $S_{i,j,k}$?

- Graphs with no subdivision of claw have been widely studied.

Tree-width of subclasses of theta-free graphs and even-hole-free graphs

Theorem (Pilipczuk, Sintiari, Thomassé, Trotignon (2020))

For $k \geq 1$, every (theta, triangle, $S_{k,k,k}$)-free graph G has tree-width at most $2(R(3, 4k-1))^3 - 1$.

Theorem (Pilipczuk, Sintiari, Thomassé, Trotignon (2020))

For $k \geq 1$, every (even hole, pyramid, K_t, $S_{k,k,k}$)-free graph G has tree-width at most $(t-1)(R(t, 4k-1))^3 - 1$.

$R_{k,s}$: Ramsey number

Idea of proof:

- Properties of graphs with high tree-width
- If graphs in the class have high tree-width, then they must contain a forbidden structure

Proof *part 1*

Essential properties of graphs with bounded tree-width:

Theorem (*Tree-width**; Pilipczuk, Sintiari, Thomassé, Trotignon (2020))

Let G be a graph and $k \in \mathbb{Z}^+$. If G does not contain:

- K_{2k}; *and*
- *an induced subgraph admitting a minimal separator of size k,*

then the tree-width of G is $\mathcal{O}\left((2k \text{ poly} \log 2k)^{19}\right)$.

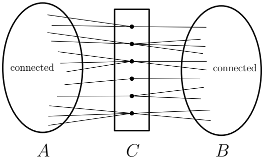


Figure: A minimal separator C separating A and B

Proof *part 1*

Suppose G s.t. $tw(G) \geq \Omega\left((2k \text{ poly} \log 2k)^{19}\right)$, then G contains a $(2k \times 2k)$-grid-minor [Chuzhoy (2016)]

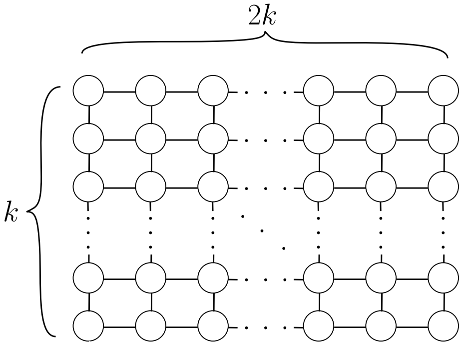


Figure: Consider the big grid minor in G with the minimum size

Proof *part 1*

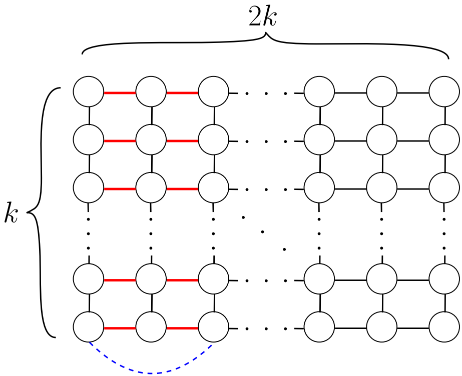


Figure: If two columns are anticomplete, then we get a minimal separator

Proof *part 1*

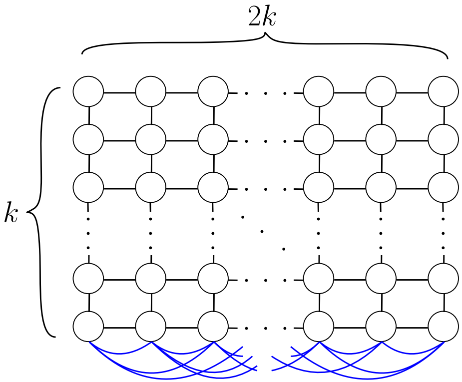


Figure: So every two columns are "adjacent"

Proof *part 1*

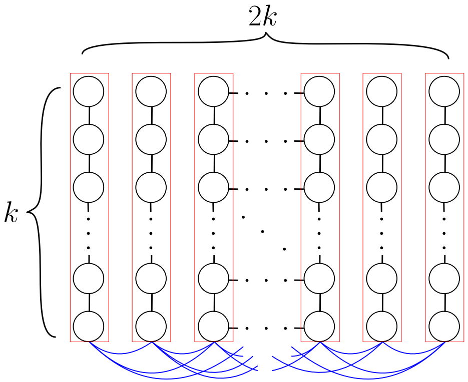


Figure: Consider every column as a component

Proof *part 1*

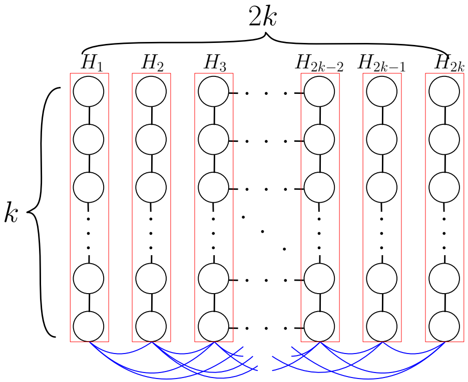


Figure: We minimize the size of the K_{2k}-model

Proof *part 1*

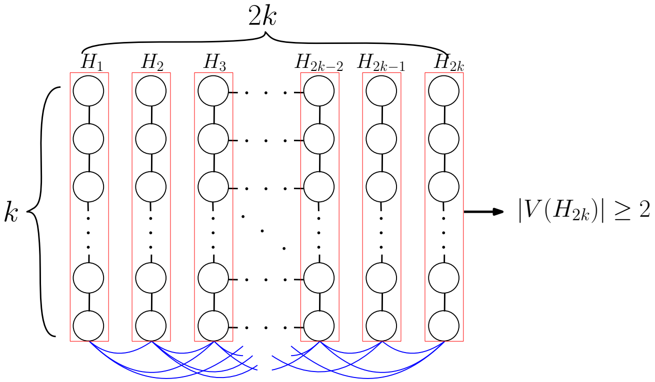


Figure: If $\forall i$, $|V(H_i)| = 1$, then we get a big clique

Proof *part 1*

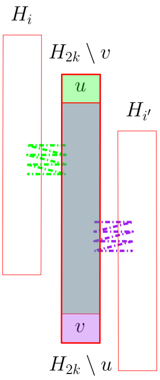


Figure: $\exists H_{2k}$ s.t. $|H_{2k}| \geq 2$; $H_{2k} \setminus v$ anticomplete to H_i

Proof *part 1*

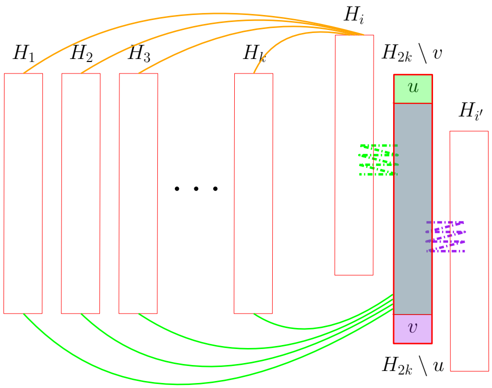


Figure: $H_{2k} \setminus v$ is connected to $\geq k$ other components

Proof *part 1*

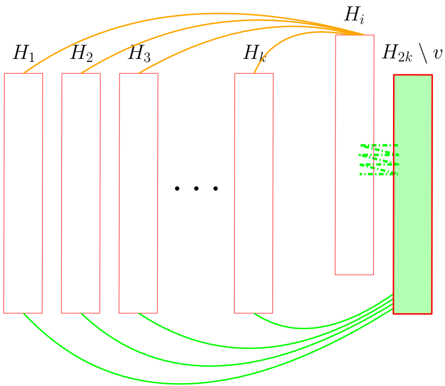


Figure: $H_{2k} \setminus v$ separated to H_i by $\geq k$ disjoint path

Proof *part 1*

A better bound for Theorem*:

Theorem (*Tree-width***; Pilipczuk, Sintiari, Thomassé, Trotignon (2020))

Let G be a graph and let $k \geq 2$ and $s \geq 1$ be positive integers. If G does not contain

- K_k
- *a minimal separator of size larger than s*

then $tw(G) \leq (k-1)s^3 - 1$.

Proof *part 2*

Recall that, we aim to prove:

Theorem

For $k \geq 1$, every (even hole, pyramid, K_t, $S_{k,k,k}$)-free graph G has tree-width at most $(t-1)(R(t, 4k-1))^3 - 1$.

Sketch of proof.

Suppose that $tw(G) > (t-1)(R(t, 4k-1))^3 - 1$.
We will prove that G contains a forbidden structure.

Proof *part 2*

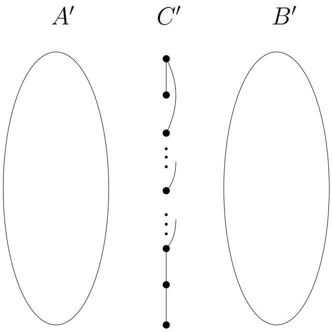


Figure: Since $tw(G)$ is large but G contains no big clique, then G contains a large minimal separator

Proof *part 2*

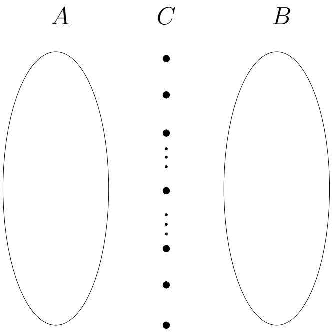


Figure: By Ramsey Thm, G contains a large minimal separator which is an independent set

Proof *part 2*

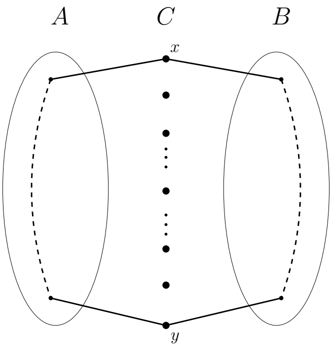


Figure: x and y have neighbors on each partition, and there exist a path connecting x and y in each of the partitions

Proof *part 2*

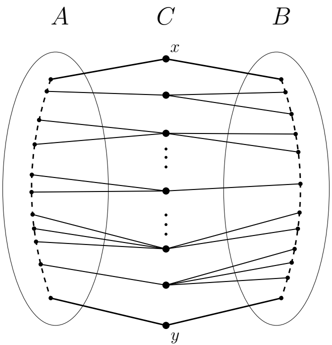


Figure: Every vertex has neighbors on each partition because C is a minimal separator

Proof *part 2*

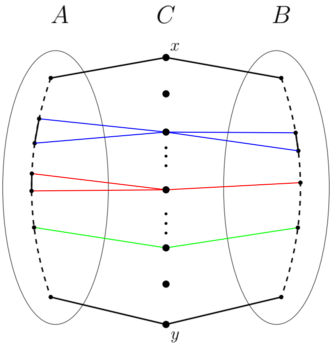


Figure: These attachments yield a forbidden structure

Proof *part 2*

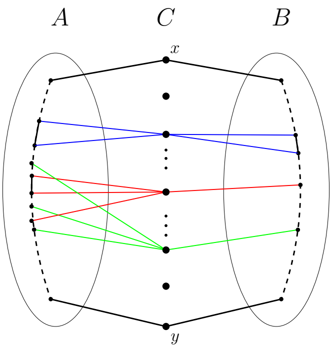


Figure: We cannot have crossing spokes because of *nestedness* property

Proof *part 2*

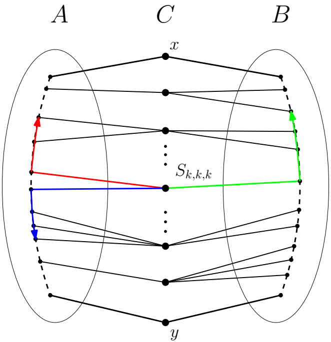


Figure: Take the middle vertex of C to start the $S_{k,k,k}$

Remark

Our result on (even hole, pyramid, K_t, $S_{i,j,k}$)-free graphs does not settle a new complexity result for the maximum independent set problem.

- MIS is polynomial in (even hole, pyramid)-free graphs [Chudnovsky, Thomassé, Trotignon, and Vušković (2019)]

Chapter 5: Bounded Maximum Degree

Motivation

What can be observed from even-hole-free layered wheels?

- Existence of large clique minor
- Existence of vertices with high degree

Are the two conditions necessary?

What is the tree-width of:

- even-hole-free graphs with no big clique minor?
- even-hole-free graphs of bounded degree?

Terminology: maximum degree of a graph (Δ)

- Degree of v ($d(v)$): the number of vertices adjacent to v
- Maximum degree:

$$\Delta(G) = \max_{v \in V(G)} d(v)$$

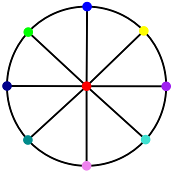


Figure: Graph G with $\Delta(G) = 8$

Even-hole-free graphs with $\Delta \leq 3$

- Subcubic $= \Delta \leq 3$
- (Theta, prism)-free graphs form superclass of EHF graphs

Theorem (*Decomposition;* Aboulker, Adler, Kim, Sintiari, Trotignon (2020))

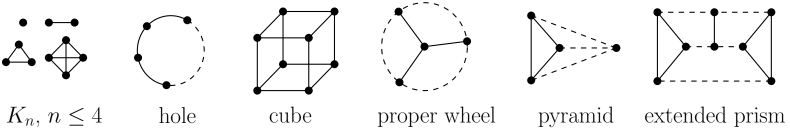
Let G be a subcubic (theta, prism)-free graph. Then one of the following holds:

- *G is a basic graph;*
- *G has a clique separator of size at most 2;*
- *G has a proper separator.*

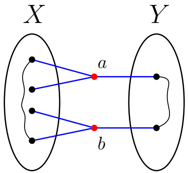
- We have a *full structure theorem* for the class of subcubic (theta, prism)-free graphs.

Even-hole-free graphs with $\Delta \leq 3$

The basic graphs:



Proper separator:



Decomposing subcubic EHF graphs (*an example*)

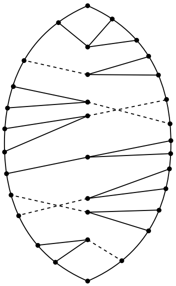


Figure: Decomposition of a non-basic subcubic (theta, prism)-free graphs

Decomposing subcubic EHF graphs (*an example*)

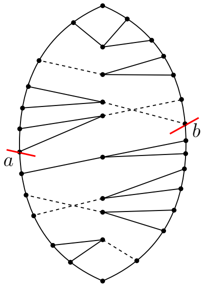


Figure: Decomposition of a non-basic subcubic (theta, prism)-free graphs

Decomposing subcubic EHF graphs (*an example*)

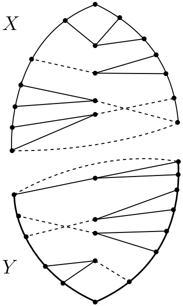


Figure: Decomposition of a non-basic subcubic (theta, prism)-free graphs

Decomposing subcubic EHF graphs (*an example*)

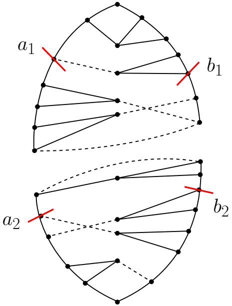


Figure: Decomposition of a non-basic subcubic (theta, prism)-free graphs

Decomposing subcubic EHF graphs (*an example*)

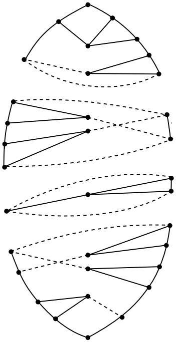


Figure: Decomposition of a non-basic subcubic (theta, prism)-free graphs

Decomposing subcubic EHF graphs (*an example*)

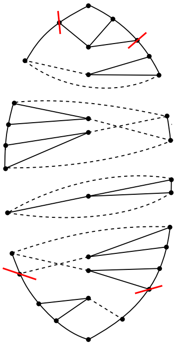


Figure: Decomposition of a non-basic subcubic (theta, prism)-free graphs

Decomposing subcubic EHF graphs (*an example*)

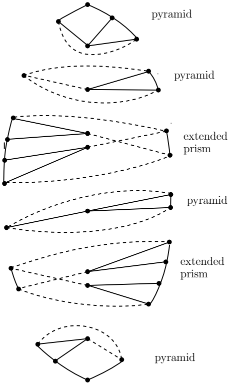


Figure: Decomposition of a non-basic subcubic (theta, prism)-free graphs

Decomposing subcubic EHF graphs (*an example*)

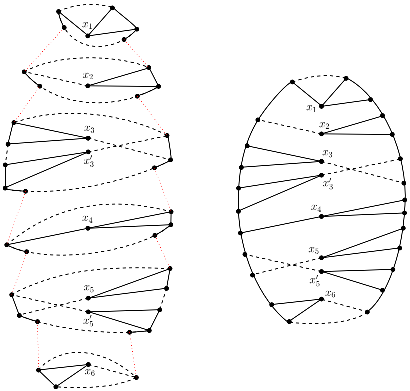


Figure: Proper gluing operations to construct a (non-basic) subcubic EHF graphs

Decomposing subcubic EHF graphs (*an example*)

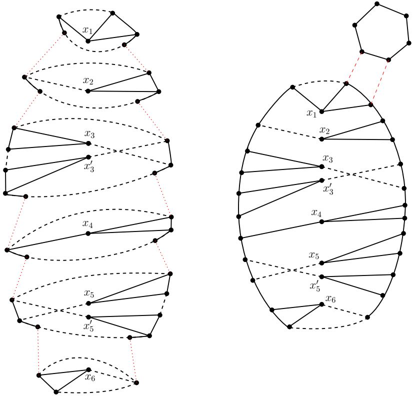


Figure: Proper gluing operations to construct a (non-basic) subcubic EHF graphs

Decomposing subcubic EHF graphs (*an example*)

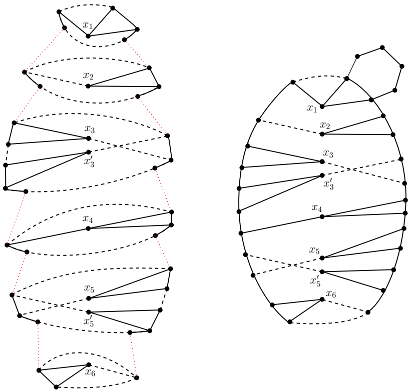


Figure: Proper gluing operations to construct a (non-basic) subcubic EHF graphs

Tree-width of subcubic EHF graphs

Theorem (*Tree-width;* Aboulker, Adler, Kim, Sintiari, Trotignon (2020))

Every subcubic (theta, prism)-free graph (and therefore every even-hole-free subcubic graph) has tree-width at most 3.

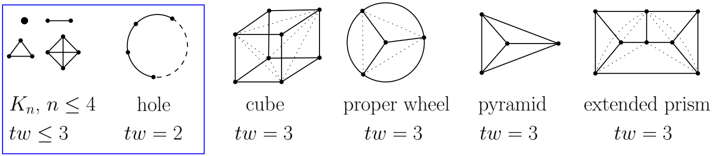


Figure: Chordal graphs containing the basic graphs

Tree-width of subcubic EHF graphs

- Gluing along a clique and proper gluing preserve the tree-width

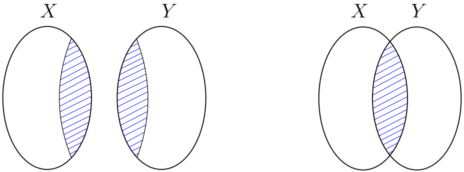


Figure: Gluing along a clique cutset

Tree-width of subcubic EHF graphs

- Gluing along a clique and proper gluing preserve the tree-width

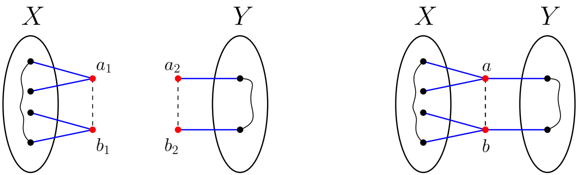


Figure: Gluing along a proper separator

Structure Theorem of EHF pyramid-free graphs $\Delta = 4$

Theorem (*Decomposition;* Sintiari, Trotignon (2020))

Let G be an (even hole, pyramid)-free graph with $\Delta(G) \leq 4$. Then one of the following holds:

- *G is a basic graph;*
- *G has a clique separator of size at most 3;*
- *G has a proper separator for $\mathcal{C}$.*

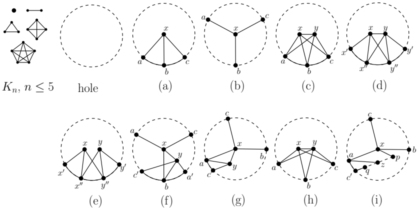


Figure: Basic graphs in the decomposition of the class

The tree-width of EHF pyramid-free graphs $\Delta = 4$

Theorem (*Tree-width;* Sintiari, Trotignon (2020))

Every (even hole, pyramid)-free graph with $\Delta \leq 4$ has tree-width at most 4.

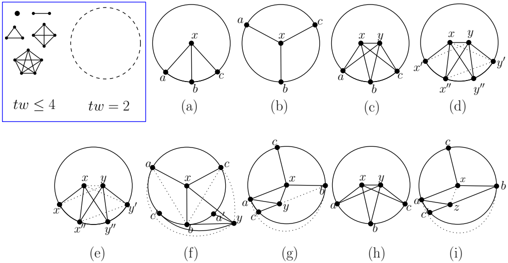


Figure: The basic graphs

- Gluing along a clique and proper gluing preserve the tree-width

Chapter 6: Conclusion and Future Works

Publication

N. L. D. Sintiari and N. Trotignon.
(Theta, triangle)-free and (even hole, K_4)-free graphs. Part 1 : Layered wheels
Published in Journal of Graph Theory (*CoRR*, abs/1906.10998), 2021.

M. Pilipczuk, S. Thomass, N. L. D. Sintiari, and N. Trotignon.
(Theta, triangle)-free and (even hole, K_4)-free graphs. Part 2 : Bounds on treewidth.
Published in Journal of Graph Theory (*CoRR*, abs/2001.01607), 2021.

P. Aboulker, I. Adler, E. J. Kim, N. L. D. Sintiari, and N. Trotignon.
On the tree-width of even-hole-free graphs.
To appear in European Journal of Combinatorics (*CoRR*, abs/2008.05504), 2021.

Future Works

Conjecture (*Logarithmic tree-width*)

G (even hole, K_4)-free, then $\text{tw}(G) = \mathcal{O}(\log |V(G)|)$.

Approach: does it exist a family $\mathcal{F}_\ell$ s.t.

- $\forall H \in \mathcal{F}_\ell$, $|V(H)| \geq r^\ell$, for some $r > 1$;
- $\forall G$ (even hole, K_4, $\mathcal{F}_\ell$)-free graph, $tw(G) \leq t \cdot \ell$ for some $t > 0$.

Open problems

Conjecture (*Grid-minor-like theorem*)

$\exists f$ *s.t. if* $\text{tw}(G) > f(k)$, *then* G *contains (as induced subgraph):*

- *a subdivision of a* $(k \times k)$*-wall; or*
- *the line graph of a subdivision of a* $(k \times k)$*-wall; or*
- *a vertex of degree at least* k.

Conjecture (*Grid-minor-like theorem (stronger version)*)

$\exists f$ *s.t. if* $\text{tw}(G) > f(k)$, *then* G *contains (as induced subgraph):*

- K_k, $K_{k,k}$; *or*
- *a subdivision of a* $(k \times k)$*-wall; or*
- *the line graph of a subdivision of a* $(k \times k)$*-wall; or*
- *a wheel with at least* k *spokes.*

Thank you for listening!